A THIRD-ORDER AND A FOURTH ORDER ITERATION PROCESS FOR NON-LINEAR EQUATIONS *

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1. Introduction. There are two paths available for the construction of high-order iteration processes for solving non-linear equations. A usual approach, and for most purposes probably the best approach is to use a recursively formed iteration function (See [1], p. 168). In this procedure, to approximate the zeros of $f(x)$ we need to use values of $f(x)$ and $f^{(1)}(x)$, but no higher derivatives. Instead, further functional substitutions are used.

An alternative is to use derivatives beyond the first. Normally this could be expected to become involved, at least for systems of many simultaneous equations. There are, however, applications for which the values of high-order derivatives are easily obtained. For such problems, the technique outlined in the next section seems of interest. Here first-order derivatives are used in usual manner to build the Newton-Raphson second-order iteration. By adding certain functions involving also second-order derivatives, we build a third-order process. By adding certain functions involving, in addition, third-order derivatives, we build a fourth-order process. A program has been written making any choice (as well as a first-order process) available for any system of equations for which the requisite derivatives can be evaluated.

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2. Procedure and Proof of Convergence. Let S be a simply-connected closed region of n-dimensional Euclidean real space R^n. Let $\vec{y} = [y_1\ y_2\ \cdots\ y_n]^t$ be an interior point of S. Let $\vec{x} = [x_1\ x_2\ \cdots\ x_n]^t$, and let $\vec{f}(\vec{x}) = [f_1(\vec{x})\ f_2(\vec{x})\ \cdots\ f_n(\vec{x})]^t$ be a vector valued function from S to R^n such that $\vec{f}(\vec{y}) = \vec{0}$. Require $\frac{\partial^4}{\partial x_r \partial x_s \partial x_t \partial x_u} \vec{f}(\vec{x})$, $1 \le r,s,t,u \le n$, to be continuous in S. We write the matrix $J(\vec{x}) = \left[\frac{\partial}{\partial x_j} f_i(\vec{x}) \right]$ where i is the row index and j the column index. We require $J(\vec{y})$ to be non-singular. We define

(1) $$\vec{g}(\vec{x}) = \vec{x} - \delta_1(\vec{x})\ J^{-1}(\vec{x})\vec{f}(\vec{x}) - \frac{1}{2}\ \delta_2(\vec{x})\ J^{-1}(\vec{x})\vec{\Psi}(\vec{x}) + \frac{1}{3!}\ \delta_3(\vec{x})\ J^{-1}(\vec{x})\vec{\Phi}(\vec{x}) - \frac{1}{2}\ \delta_4(\vec{x})\ J^{-1}(\vec{x})\vec{\Gamma}(\vec{x}).$$

Here

$$\vec{\Psi}(\vec{x}) = [\Psi_1(\vec{x})\ \Psi_2(\vec{x})\ \cdots\ \Psi_n(\vec{x})]^t$$

and

$$\Psi_i(\vec{x}) = \vec{f}(\vec{x})^t J^{-1}(\vec{x})^t\ D_i(\vec{x})\ J^{-1}(\vec{x})\ \vec{f}(\vec{x})$$

while

$$D_i(\vec{x}) = \left[\frac{\partial^2}{\partial x_r \partial x_s} f_i(\vec{x}) \right] .$$

Here r is the row index, and s is the column index.

Also,

$$\vec{\Phi}(\vec{x}) = [\vec{f}(\vec{x})^t J^{-1}(\vec{x})^t\ {}^{1}\vec{\Phi}(\vec{x})\quad \vec{f}(\vec{x})^t J^{-1}(\vec{x})^t\ {}^{2}\vec{\Phi}(\vec{x})\ \cdots\ \vec{f}(\vec{x})^t J^{-1}(\vec{x})^t\ {}^{n}\vec{\Phi}(\vec{x})]^t$$

where

$${}^{s}\vec{\Phi}(\vec{x}) = [{}^{1s}\Phi(\vec{x})\quad {}^{2s}\Phi(\vec{x})\ \cdots\ {}^{ns}\Phi(\vec{x})]^t ,$$

while

$$^{rs}\Phi(\vec{x}) = \vec{f}(\vec{x})^t J^{-1}(\vec{x})^t \; ^{rs}C(\vec{x}) \; J^{-1}(\vec{x})\vec{f}(\vec{x})$$

where

$$^{rs}C(\vec{x}) = \left[\frac{\partial^3}{\partial x_r \partial x_i \partial x_j} f_s(\vec{x})\right] = \left[^{rs}c_{ij}(\vec{x})\right] .$$

Here i is the row index, and j is the column index.

Moreover

$$\vec{\Gamma}(\vec{x}) = [\gamma_1(\vec{x}) \; \gamma_2(\vec{x}) \; \cdots \; \gamma_n(\vec{x})]^t$$

where

$$\gamma_i(\vec{x}) = \vec{f}(\vec{x})^t \; J^{-1}(\vec{x})^t \; D_i(\vec{x}) \; J^{-1}(\vec{x}) \; \vec{\Psi}(\vec{x}).$$

Let $\delta_1(\vec{x})$, $\delta_2(\vec{x})$, $\delta_3(\vec{x})$, and $\delta_4(\vec{x})$ be real valued functions such that $0 < \delta_1(\vec{x}) < 2$ and $\delta_i(\vec{x})$, $i = 2, 3, 4$, are arbitrary.

Define an iteration process by

$$\vec{x}^{(k+1)} = \vec{g}(\vec{x}^{(k)}). \tag{2}$$

Then if $\vec{x}^{(1)}$ is "near enough" $\vec{y}$, it is true that $\lim_{k\to\infty} \vec{x}^{(k)} = \vec{y}$.

Moreover, if $0 < \delta_1(x) < 2$ and $\delta_i(x)$, $i = 2, 3, 4$, are arbitrary, the process is linear. If $\delta_1(x) = 1$, the process is quadratic. If $\delta_1(x) = \delta_2(x) = 1$, the process is cubic. If $\delta_i(x) = 1$, $i = 1, 2, 3, 4$, the process is fourth order.

<u>PROOF</u>

By Taylor's Theorem,

$$\vec{f}(\vec{y}) = \vec{f}(\vec{x}) + \sum_{i=1}^{n} (y_i - x_i) \frac{\partial}{\partial x_i} \vec{f}(\vec{x}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (y_i - x_i)(y_j - x_j) \frac{\partial^2}{\partial x_i \partial x_j} \vec{f}(\vec{x}) \tag{3}$$

$$+ \frac{1}{3!} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (y_i - x_i)(y_j - x_j)(y_k - x_k) \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} \vec{f}(\vec{x})$$

$$+ \frac{1}{4!} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} (y_i - x_i)(y_j - x_j)(y_k - x_k)$$
$$(y_m - x_m)\ \vec{h}_{ijkm}(\vec{x})$$

where $\vec{h}_{ijkm}(\vec{x})$ is given by

$$\left[\frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_m} f_1(\vec{x} + \Theta_1(\vec{y} - \vec{x}))\ \frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_m} f_2(\vec{x} + \Theta_2(\vec{y} - \vec{x})) \right.$$
$$\left. \cdots\ \frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_m} f_n(\vec{x} + \Theta_n(\vec{y} - \vec{x})) \right]^t,\ 0 < \Theta_i < 1.$$

Note that

$$\vec{f}(\vec{y}) = \vec{0},\quad \sum_{i=1}^{n} (y_i - x_i) \frac{\partial}{\partial x_i} \vec{f}(\vec{x}) = - J(\vec{x})\ (\vec{x} - \vec{y}),$$

and $\sum_{i=1}^{n} \sum_{j=1}^{n} (y_i - x_i)(y_j - x_j) \frac{\partial^2}{\partial x_i \partial x_j} \vec{f}(\vec{x})$ is

$$[(\vec{x} - \vec{y})^t D_1(\vec{x})(\vec{x} - \vec{y})\ (\vec{x} - \vec{y})^t D_2(\vec{x})(\vec{x} - \vec{y})\ \cdots\ (\vec{x} - \vec{y})^t D_n(\vec{x})(\vec{x} - \vec{y})]^t$$

where, as above, $D_i(x) = \left[\frac{\partial^2}{\partial x_r \partial x_s} f_i(\vec{x}) \right]$.

Let

$$\vec{L}(\vec{x}) = [(\vec{x} - \vec{y})^t D_1(\vec{x})(\vec{x} - \vec{y})\ (\vec{x} - \vec{y})^t D_2(\vec{x})(\vec{x} - \vec{y})\ \cdots$$
$$(\vec{x} - \vec{y})^t D_n(\vec{x})(\vec{x} - \vec{y})]^t$$

and

$$\vec{R}_1(\vec{x}) = - \frac{1}{4!} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} (x_i - y_i)(x_j - y_j)(x_k - y_k)$$
$$(x_m - y_m)\vec{h}_{ijkm}(\vec{x}).$$

Then (3) becomes

$$(4) \qquad \vec{f}(\vec{x}) = J(\vec{x})(\vec{x} - \vec{y}) - \frac{1}{2}\vec{L}(\vec{x})$$

$$+ \frac{1}{3!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_i - y_i)(x_j - y_j)(x_k - y_k)$$

$$\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\vec{f}(\vec{x}) + \vec{R}_1(\vec{x})$$

where $\vec{R}_1(\vec{x})$ is a vector made up of terms of fourth order in the $x_i - y_i$.

If we take the second term on the right side of (1) and substitute (4)

$$(5) \qquad \delta_1(\vec{x})J^{-1}(\vec{x})\vec{f}(\vec{x}) = \delta_1(\vec{x})\{(\vec{x} - \vec{y}) - \frac{1}{2}J^{-1}(\vec{x})\vec{L}(\vec{x})$$

$$+ \frac{1}{3!}J^{-1}(\vec{x})\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_i - y_i)(x_j - y_j)$$

$$(x_k - y_k)\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\vec{f}(\vec{x})\} + \vec{R}_{11}(\vec{x})$$

where $\vec{R}_{11}(\vec{x})$ contains only terms of fourth degree in the $x_i - y_i$.

If we take the third term on the right side of (1) and substitute (4)

$$(6) \qquad \frac{1}{2}\delta_2(\vec{x})J^{-1}(\vec{x})\vec{\Psi}(\vec{x}) = \frac{1}{2}\delta_2(\vec{x})J^{-1}(\vec{x})\{\vec{L}(\vec{x})$$

$$- [(\vec{x} - \vec{y})^t D_1(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x})$$

$$(\vec{x} - \vec{y})^t D_2(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x})$$

$$\cdots (\vec{x} - \vec{y})^t D_n(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x})]^t\}$$

$$+ \vec{R}_{12}(\vec{x})$$

where $\vec{R}_{12}(\vec{x})$ contains only terms of fourth and higher degree in the $x_i - y_i$.

If we take the fourth term on the right of (1) and substitute (4)

$$(7)\qquad \frac{1}{3!}\delta_3(\vec{x})J^{-1}(\vec{x})\vec{\Phi}(\vec{x}) = \frac{1}{3!}\delta_3(\vec{x})J^{-1}(\vec{x}) \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_i - y_i)(x_j - y_j)(x_k - y_k)\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\vec{f}(\vec{x}) + \vec{R}_{13}(\vec{x})$$

where $\vec{R}_{13}(\vec{x})$ contains only terms of fourth and higher degree in the $x_i - y_i$.

If we take the fifth term on the right side of (1) and substitute (4)

$$(8)\qquad \frac{1}{2}\delta_4(\vec{x})J^{-1}(\vec{x})\vec{\Gamma}(\vec{x}) = \frac{1}{2}\delta_4(\vec{x})J^{-1}(\vec{x}) [(\vec{x} - \vec{y})^t D_1(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x}) \quad (\vec{x} - \vec{y})^t D_2(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x}) \quad \cdots \quad (\vec{x} - \vec{y})^t D_n(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x})]^t + \vec{R}_{14}(\vec{x})$$

where $\vec{R}_{14}(\vec{x})$ contains only terms of fourth and higher degree in the $x_i - y_i$.

Combining (1), (5), (6), (7), and (8) we obtain

$$(9)\qquad \vec{g}(\vec{x}) - \vec{y} = (1 - \delta_1(\vec{x}))(\vec{x} - \vec{y}) + J^{-1}(\vec{x})\{\frac{1}{2}(\delta_1(\vec{x}) - \delta_2(\vec{x}))\vec{L}(\vec{x}) + \frac{1}{3!}(\delta_3(\vec{x}) - \delta_1(\vec{x})) \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_i - y_i)(x_j - y_j)(x_k - y_k)\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\vec{f}(\vec{x}) + \frac{1}{2}(\delta_2(\vec{x}) - \delta_4(\vec{x}))[(\vec{x} - \vec{y})^t D_1(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x}) \quad (\vec{x} - \vec{y})^t D_2(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x}) \quad \cdots \quad (\vec{x} - \vec{y})^t D_n(\vec{x})J^{-1}(\vec{x})\vec{L}(\vec{x})]^t + \vec{R}(\vec{x})$$

where

$$\vec{R}(\vec{x}) = -\{\vec{R}_{11}(\vec{x}) + \vec{R}_{12}(\vec{x}) + \vec{R}_{13}(\vec{x}) + \vec{R}_{14}(\vec{x})\}$$

whose entries contain only terms of fourth and higher degree in the $x_i - y_i$. If we choose $\delta_i(x) = 1$, $i = 1, 2, 3, 4$, (9) becomes

(10) $$\vec{g}(\vec{x}) - \vec{y} = \vec{R}(\vec{x}).$$

If $J(\vec{y})$ is not singular, there exists a closed region S_1 having $\vec{y}$ as an interior point such that, for $\vec{x} \in S_1$, $J(x)$ is not singular, and the coefficients of the terms of the form $(x_i - y_i)(x_j - y_j)(x_k - y_k)(x_m - y_m)$ in $\vec{R}(\vec{x})$ are bounded. If, for a column vector $\vec{z}$ with elements z_i, by $||\vec{z}||$ we mean $\max_i |z_i|$, it follows that there is a positive constant M such that for $\vec{x} \in S_1$

(11) $$||R(\vec{x})|| \le M||\vec{x} - \vec{y}||^4 .$$

Now let $\vec{x}^{(1)}$ be so chosen that $||\vec{x}^{(1)} - \vec{y}|| < \Theta/M^{1/3}$ where $0 < \Theta < 1$ and $\vec{x}^{(1)}$ lies in S_1. For $\vec{x}^{(1)} \in S_1$, using (2) and (10)

$$\vec{x}^{(2)} - \vec{y} = \vec{R}(\vec{x}^{(1)})$$

so that

$$||\vec{x}^{(2)} - \vec{y}|| \le M||\vec{x}^{(1)} - \vec{y}||^4 < \Theta^4/M^{1/3}$$

and $\vec{x}^{(2)} \in S_1$ since $||\vec{x}^{(2)} - \vec{y}|| < \Theta/M^{1/3}$.

Inductively it is seen that

$$||\vec{x}^{(k+1)} - \vec{y}|| \le M||\vec{x}^{(k)} - \vec{y}||^4 < \Theta^{4^k}/M^{1/3}$$

and $\vec{x}^{(k+1)} \in S_1$.

Thus $\lim_{k\to\infty} \vec{x}^{(k)} = \vec{y}$ and (1) is a fourth order process.

If $0 < \delta_1(\vec{x}) < 2$ and $\delta_i(\vec{x})$, $i = 2, 3, 4$, are arbitrary, (9) becomes

$$\vec{g}\,(\vec{x}) - \vec{y} = (1 - \delta_1(\vec{x}))\ (\vec{x} - \vec{y}) + \vec{T}_1(\vec{x})$$

where $\vec{T}_1(\vec{x})$ contains only terms of second and higher degree in the

$x_i - y_i$. In a manner similar to that above, (1) can be shown to be a first order process.

If $\delta_1(\vec{x}) = 1$ and $\delta_i(\vec{x})$, $i = 2, 3, 4$, are arbitrary, (9) becomes

$$\vec{g}\,(\vec{x}) - \vec{y} = \vec{T}_2(\vec{x})$$

where $\vec{T}_2(\vec{x})$ contains only terms of second and higher degree in the $x_i - y_i$. In a manner similar to that above, (1) can be shown to be a second order process.

If $\delta_1(\vec{x}) = \delta_2(\vec{x}) = 1$ with $\delta_3(\vec{x})$ and $\delta_4(\vec{x})$ being arbitrary, (9) becomes

$$\vec{g}\,(\vec{x}) - \vec{y} = \vec{T}_3(\vec{x})$$

where $\vec{T}_3(\vec{x})$ contains only terms of third and higher degree in the $x_i - y_i$. In a manner similar to that above, (1) can be shown to be a third order process.

REFERENCES

[1] J. F. Traub, *Iterative methods for the solution of equations*, Prentice-Hall Inc., New Jersey, 1964.